

Vacuum instability in slowly varying electric fields

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Abstract

Nonperturbative methods are well-developed for QED with the so-called t -electric potential steps. In this case a calculation techniques is based on the existence of specific exact solutions (in- and out-solutions) of the Dirac equation. However, there are only few cases when such solutions are known. Here, we demonstrate that for t -electric potential steps slowly varying with time there exist physically reasonable approximations that maintain nonperturbative character of QED calculations even in the absence of the exact solutions. Defining the slowly varying regime in general terms, we can observe an universal character of vacuum effects caused by strong electric field. In the present article, we find universal approximate representations for the total density of created pairs and vacuum mean values of the current density and energy-momentum tensor that hold true for arbitrary t -electric potential step slowly varying with time. These representations do not require knowledge of corresponding solutions of the Dirac equation, they have a form of simple functionals of a given slowly varying electric field. We establish relations of these representations with leading terms of derivative expansion approximation. These results allow one to formulate some semiclassical approximations that are not restricted by smallness of differential mean numbers of created pairs.

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I. INTRODUCTION

It is well-known that in QED with strong electric-like external fields there exists so-called vacuum instability due to real particle creation caused by the external field. A number of publications, reviews and books are devoted to the effect of particle creation itself and to developing different nonperturbative calculation methods in theories with unstable vacuum: analytical (semiclassical and based on exact solutions) and numerical, see Refs. [1–3] for a review. Most semiclassical and numerical methods are applied to Schwinger’s effective action and related formulas [4] (see Ref. [5] for a review), to calculate the probability for a vacuum to remain a vacuum. They are well-grounded for not very strong electric fields, when the probability of a pair creation is exponentially small. There exists the derivative expansion approximation method, which being applied to Schwinger’s effective action, allows one to treat effectively slowly varying strong fields [6, 7]. However, it should be noted that the probability for a vacuum to remain a vacuum is not very informative for studying the time evolution of vacuum effects caused by strong electric field. It can be seen that in some situations in astrophysics and condensed matter the time evolution of vacuum effects caused by strong electric field is of significant interest, e.g., see Refs. [2, 8–11]. In the case of strong external fields, nonperturbative methods are well-developed for QED with two specific configurations of external backgrounds, namely for the so-called t -electric potential steps [3, 12, 13] and x -electric potential steps [14]. In both cases the calculation technics is based on the existence of specific exact solutions (in- and out-solutions) of the Dirac equation. Under this condition, all the probability amplitudes and mean values in the backgrounds under consideration have some nonperturbative integral representations via these in- and out-solutions. At present, there are known only few types of t - and x -electric potential steps when such solutions are known, we call these cases exactly solvable cases. In QED with t -electric potential steps, exactly solvable cases, that have real physical importance, are Sauter-like electric field, the so-called T -constant electric field (a uniform electric field which acts during a finite time interval T , this case includes the constant electric field when $T \rightarrow \infty$), and exponentially growing and decaying electric fields. Using the corresponding exact solutions, different characteristics of quantum processes related to the particle creation were calculated in detail, see [6, 8, 15–22], respectively. And here we come to the question if there exist physically reasonable approximations in QED with the above described strong

backgrounds that maintain nonperturbative character of calculations and allow one to go beyond dealing with the existence of the exact solutions? In this article, we demonstrate that such a possibility exist in the case of QED with the t -electric potential steps slowly varying with time, similar possibility in the case of QED with x -electric potential steps will be presented in our next publication.

In Section II, we give a definition of slowly varying t -electric potential steps and revised vacuum instability due to such backgrounds for the existing exactly solvable cases. In Section III, we stress universal features of the vacuum instability in these examples. We derive universal approximate representations for the total density of created pairs and vacuum mean values of current density and energy-momentum tensor (EMT) components that hold true for arbitrary t -electric potential step slowly varying with time. These representations do not require knowledge of corresponding solutions of the Dirac equation, they have a form of simple functionals of a given slowly varying electric field. We establish relations of these representations with leading terms of derivative expansion approximation. These results allow one to formulate some semiclassical approximations that are not restricted by smallness of differential mean numbers of created pairs. In the Appendix A, we briefly describe a nonperturbative formulation of QED with t -electric potential steps.

II. SLOWLY VARYING t -ELECTRIC POTENTIAL STEPS, EXACTLY SOLVABLE CASES

We call $E(t)$ a slowly varying electric field on a time interval Δt if the following condition holds true:

$$\left| \frac{\overline{\dot{E}(t)}\Delta t}{\overline{E(t)}} \right| \ll 1, \quad \Delta t/\Delta t_{\text{st}}^m \gg 1, \quad (2.1)$$

where $\overline{E(t)}$ and $\overline{\dot{E}(t)}$ are mean values of $E(t)$ and $\dot{E}(t)$ on the time interval Δt , respectively, and Δt is significantly higher than the time scale Δt_{st}^m which is

$$\Delta t_{\text{st}}^m = \Delta t_{\text{st}} \max \left\{ 1, m^2/e\overline{E(t)} \right\}, \quad \Delta t_{\text{st}} = \left[e\overline{E(t)} \right]^{-1/2}. \quad (2.2)$$

We are primarily interested in strong electric fields, $m^2/e\overline{E(t)} \lesssim 1$. In this case, inequality (2.2) is simplified to the form $\Delta t/\Delta t_{\text{st}} \gg 1$, in which the mass m is absent. In such cases, the potential of the corresponding electric steps hardly differs from the potential of a constant

electric field,

$$U(t) = -eA_x(t) \approx U_c(t) = e\overline{E(t)}t + U_0, \quad (2.3)$$

on the time interval Δt , where U_0 is a given constant. This behavior is inherent for the fields of known exact solvable cases with appropriate parameters, namely: the peak field, the T -constant electric field, and Sauter-like electric field.

The complete sets of solutions of the Dirac equation, given by Eq. (A4), are determined by the functions ${}_z\varphi_n(t)$ and ${}_{\bar{z}}\varphi_n(t)$. These functions are known explicitly for the following electric fields.

The Sauter-like (or adiabatic or pulse) electric field and its vector potential have the form

$$E(t) = E_0 \cosh^{-2}(t/T_S), \quad A_x(t) = -T_S E_0 \tanh(t/T_S). \quad (2.4)$$

where the parameter $T_S > 0$ sets time scale. The functions ${}_z\varphi_n(t)$ and ${}_{\bar{z}}\varphi_n(t)$ and the number of created pairs N_n^{cr} are given, for example, in Ref. [16]. We have the case of slowly varying field if

$$\sqrt{eE_0}T_S \gg \max\left(1, m/\sqrt{eE_0}\right). \quad (2.5)$$

In this case, the leading contribution to the total number of pairs created from vacuum is formed in the range of $|p_x| < eE_0T_S$ and small $\pi_\perp \ll eE_0T_S$. In this range the differential mean number of created pairs have approximately the following form

$$N_n^{\text{cr}} \approx N_n^{\text{as}} = \exp\{-\pi T_S [p_0(+\infty) + p_0(-\infty) - 2eE_0T_S]\}, \quad (2.6)$$

where $p_0(\pm\infty)$ are the energies given by Eq. (A3) in Appendix A. This distribution has a maximum at $p_x = 0$. This maximum coincides with differential number of created pairs in a constant electric field [1, 17],

$$N_n^{\text{cr}} \approx N_n^0 = e^{-\pi\lambda_0}, \quad \lambda_0 = \frac{\pi_\perp^2}{eE_0}, \quad \pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2}. \quad (2.7)$$

The so-called T -constant electric field does not change within the time interval T and is zero outside of it,

$$E(t) = \begin{cases} 0, & t \in \text{I} \\ E_0, & t \in \text{II} \\ 0, & t \in \text{III} \end{cases} \implies A_x(t) = \begin{cases} -E_0 t_{\text{in}}, & t \in \text{I} \\ -E_0 t, & t \in \text{II} \\ -E_0 t_{\text{out}}, & t \in \text{III} \end{cases}, \quad (2.8)$$

where I denotes the in-region $t \in (-\infty, t_{\text{in}}]$, II is the intermediate region where the electric field is nonzero $t \in (t_{\text{in}}, t_{\text{out}})$ and III is the out-region $t \in [t_{\text{out}}, +\infty)$ and $t_{\text{out}}, t_{\text{in}}$ are constants, $t_{\text{out}} - t_{\text{in}} = T$. We choose $t_{\text{out}} = -t_{\text{in}} = T/2$. The functions ${}_{-}\varphi_n(t)$ and ${}_{+}\varphi_n(t)$ and the distribution N_n^{cr} are found in Ref. [16]. The T -constant field can be considered as slowly varying if

$$\sqrt{eE_0}T \gg \max(1, m^2/eE_0). \quad (2.9)$$

In this case, the leading contribution to the total number of pairs created is formed in the range of $|p_x| < eE_0T/2$ and small $\pi_{\perp} \ll eE_0T/2$ and has a form (2.7).

A peak electric field $E(t)$ is composed of two parts, one of them is increasing exponentially on the time-interval I = $(-\infty, 0]$, and reaches its maximal magnitude $E_0 > 0$ at the end of the interval $t = 0$, the second part decreases exponentially on the time-interval II = $(0, +\infty)$ having at $t = 0$ the same magnitude E_0 . The vector potential $A_x(t)$ and the field $E_x(t)$ are

$$E(t) = E_0 \begin{cases} e^{k_1 t}, & t \in \text{I} \\ e^{-k_2 t}, & t \in \text{II} \end{cases}, \quad A_x(t) = E_0 \begin{cases} k_1^{-1}(-e^{k_1 t} + 1), & t \in \text{I} \\ k_2^{-1}(e^{-k_2 t} - 1), & t \in \text{II} \end{cases}, \quad (2.10)$$

where k_1 and k_2 are positive constants. The functions ${}_{\zeta}\varphi_n(t)$ and ${}^{\zeta}\varphi_n(t)$ and the distribution N_n^{cr} are found in Ref. [22]. In particular, in the intervals I and II we have the following behavior

$$\begin{aligned} {}_{+}\varphi_n(t) &= {}_{+}\mathcal{N} \exp(i\pi\nu_1/2) y_2^1(\eta_1), \quad {}_{-}\varphi_n(t) = {}_{-}\mathcal{N} \exp(-i\pi\nu_1/2) y_1^1(\eta_1), \quad t \in \text{I}; \\ {}_{+}\varphi_n(t) &= {}_{+}\mathcal{N} \exp(-i\pi\nu_2/2) y_1^2(\eta_2), \quad {}_{-}\varphi_n(t) = {}_{-}\mathcal{N} \exp(i\pi\nu_2/2) y_2^2(\eta_2), \quad t \in \text{II}. \\ y_1^j(\eta_j) &= e^{-\eta_j/2} \eta_j^{\nu_j} \Phi(a_j, c_j; \eta_j), \quad y_2^j(\eta_j) = e^{\eta_j/2} \eta_j^{-\nu_j} \Phi(1 - a_j, 2 - c_j; -\eta_j), \end{aligned} \quad (2.11)$$

where $\Phi(a, c; \eta)$ is confluent hypergeometric function [24] and

$$\begin{aligned} \eta_1 &= ih_1 e^{k_1 t}, \quad \eta_2 = ih_2 e^{-k_2 t}, \quad h_j = 2eE_0 k_j^{-2}, \quad j = 1, 2, \\ c_j &= 1 + 2\nu_j, \quad a_j = \frac{1}{2}(1 + \chi) + (-1)^j \frac{i\pi_j}{k_j} + \nu_j, \\ \nu_j &= \frac{i\omega_j}{k_j}, \quad \omega_j = \sqrt{\pi_j^2 + \pi_{\perp}^2}, \quad \pi_j = p_x - (-1)^j \frac{eE_0}{k_j}. \end{aligned}$$

Slowly varying peak field corresponds to small values of k_1 and k_2 ,

$$\min(h_1, h_2) \gg \max(1, m^2/eE_0). \quad (2.12)$$

In this case, the main contributions to N_n^{cr} are formed in the ranges $\pi_\perp < \pi_1 \leq eE_0/k_1$ and $-eE_0/k_2 < \pi_2 < -\pi_\perp$, where they have the following forms:

$$\begin{aligned} N_n^{\text{cr}} &\approx \exp \left[-\frac{2\pi}{k_1} (\omega_1 - \pi_1) \right], \quad \pi_\perp < \pi_1 \leq eE_0/k_1, \\ N_n^{\text{cr}} &\approx \exp \left[-\frac{2\pi}{k_2} (\omega_2 + \pi_2) \right], \quad -eE_0/k_2 < \pi_2 < -\pi_\perp. \end{aligned} \quad (2.13)$$

In the examples under discussion, increasing and decreasing components of the fields are almost symmetric. We have an essentially asymmetric configuration in the case of the peak field, when the field switches abruptly on at $t = 0$, that is, k_1 is sufficiently large,

$$eE_0 k_1^{-2} \ll 1, \quad \omega_1/k_1 \ll 1, \quad (2.14)$$

while the parameter $k_2 > 0$ is arbitrary and includes the case of a smooth switching off. It is the so-called exponentially decaying electric field, see Ref. [22] for details. The case of slowly varying field we have when

$$h_2 \gg \max(1, m^2/eE_0).$$

In this case, the leading contribution to the total number of pairs created from vacuum is formed in the range $-eE_0/k_2 < \pi_2 < -\pi_\perp$. In this range, N_n^{cr} coincides with the form given by second line in Eq.(2.13). Note, that due to the invariance of the mean numbers N_n^{cr} under the simultaneous change $k_1 \leftrightarrow k_2$ and $\pi_1 \leftrightarrow -\pi_2$, one can easily transform this situation to the case with a large k_2 and arbitrary $k_1 > 0$.

As it follows from calculations in the exactly solvable cases, for slowly varying electric fields differential mean numbers of electron-positron pairs created from the vacuum N_n^{cr} are quasiconstant over the wide range of the longitudinal momentum p_x for any given transversal momenta, although these distributions N_n^{cr} are different for different field configurations. Furthermore, in all these cases, there exist wide subranges, in which these distributions N_n^{cr} coincide with the corresponding distributions N_n^0 in a constant electric field, given by Eq. (2.7). We call this phenomenon a stabilization of the particle creation effect. In these subranges the mean numbers N_n^{cr} hardly depend of the details of switching on and off of electric field.

The total number of pairs created from a vacuum by an uniform electric field is proportional to the space volume $V_{(d-1)}$ as $N^{\text{cr}} = V_{(d-1)} n^{\text{cr}}$, and the corresponding densities n^{cr}

have the form

$$n^{\text{cr}} = \frac{J_{(d)}}{(2\pi)^{d-1}} \int d\mathbf{p} N_n^{\text{cr}}. \quad (2.15)$$

In deriving Eq. (2.15) a sum over all momenta \mathbf{p} was transformed into an integral and summation over spin projections was fulfilled, $J_{(d)} = 2^{[d/2]-1}$. In slowly varying fields, total increment of the longitudinal kinetic momentum, which is $\Delta U = e|A_x(+\infty) - A_x(-\infty)|$, is large and can be used as large parameter. Then the integral in the right hand side of Eq. (2.15) can be approximated by an integral over a subrange Ω that gives the dominant contribution with respect to the total increment to the mean number of created particles,

$$\Omega : n^{\text{cr}} \approx \tilde{n}^{\text{cr}} = \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{\mathbf{p} \in \Omega} d\mathbf{p} N_n^{\text{cr}}. \quad (2.16)$$

The dominant contributions \tilde{n}^{cr} are proportional to increments of the longitudinal kinetic momentum, which, in general, are different for different fields and, for example, have the following forms in the exactly solvable cases:

$$\begin{aligned} \Delta U_{\text{p}} &= eE_0 (k_1^{-1} + k_2^{-1}) \text{ for a peak field,} \\ \Delta U_{\text{T}} &= eE_0 T \text{ for } T\text{-const. field,} \\ \Delta U_{\text{S}} &= 2eE_0 T_{\text{S}} \text{ for Sauter-like fields.} \end{aligned} \quad (2.17)$$

We note that ΔU_{p} in Eq. (2.17) corresponds to the case of an exponentially decaying field at $k_1^{-1} \rightarrow 0$.

In terms of the introduced quantities (2.17), the densities \tilde{n}^{cr} in the exactly solvable cases under consideration have the following forms¹ [16, 22] (see [23] for more details):

$$\begin{aligned} \tilde{n}^{\text{cr}} &= r^{\text{cr}} \frac{\Delta U_{\text{p}}}{eE_0} G\left(\frac{d}{2}, \pi \frac{m^2}{eE_0}\right) \text{ for a peak field,} \\ \tilde{n}^{\text{cr}} &= r^{\text{cr}} \frac{\Delta U_{\text{T}}}{eE_0} \text{ for T-const. field,} \\ \tilde{n}^{\text{cr}} &= r^{\text{cr}} \frac{\Delta U_{\text{S}}}{2eE_0} \delta \text{ for Sauter-like fields,} \end{aligned} \quad (2.18)$$

where

$$\begin{aligned} r^{\text{cr}} &= \frac{J_{(d)} (eE_0)^{d/2}}{(2\pi)^{d-1}} \exp\left\{-\pi \frac{m^2}{eE_0}\right\}, \quad G(\alpha, x) = \int_1^\infty \frac{ds}{s^{\alpha+1}} e^{-x(s-1)} = e^x x^\alpha \Gamma(-\alpha, x), \\ \delta &= \int_0^\infty dt t^{-1/2} (t+1)^{-(d+1)/2} \exp\left(-t\pi \frac{m^2}{eE_0}\right) = \sqrt{\pi} \Psi\left(\frac{1}{2}, \frac{2-d}{2}; \pi \frac{m^2}{eE_0}\right). \end{aligned} \quad (2.19)$$

¹ Note that the derivation of total quantities for the Sauter-like case in Refs. [16] is given for $\lambda_0 > 1$.

However, the final form of $\delta = \sqrt{\pi} \Psi\left(\frac{1}{2}, \frac{2-d}{2}; \pi \frac{m^2}{eE_0}\right)$ is given correctly for arbitrary m^2/eE_0 .

Here $\Gamma(-\alpha, x)$ is the incomplete gamma function and $\Psi(a, b; x)$ is the confluent hypergeometric function [24]. Equating the densities n^{cr} for Sauter-like field and for the peak field to the density n^{cr} for the T -constant field, we find an effective time T_{eff} of the field duration in both cases,

$$\begin{aligned} T_{\text{eff}} &= T_{\text{S}}\delta \text{ for Sauter-like fields,} \\ T_{\text{eff}} &= (k_1^{-1} + k_2^{-1}) G\left(\frac{d}{2}, \pi \frac{m^2}{eE_0}\right) \text{ for a peak field.} \end{aligned} \quad (2.20)$$

Note that the effective time T_{eff} for an exponentially decaying field is given by the second line in Eq. (2.20) as $k_1^{-1} \rightarrow 0$. By the definition $T_{\text{eff}} = T$ for the T -constant field. One can say that the Sauter-like, the peak electric fields, and the exponentially decaying field with the same T_{eff} are equivalent to the T -constant field with respect to the pair production. Note that the factors G and δ in Eq. (2.19) for a weak electric field ($m^2/eE_0 \gg 1$) and for a strong enough electric field ($m^2/eE_0 \ll 1$) can be approximated as

$$\begin{aligned} G\left(\frac{d}{2}, \pi \frac{m^2}{eE_0}\right) &\approx \frac{eE_0}{\pi m^2}, \quad \delta \approx \frac{\sqrt{eE_0}}{m}, \quad \frac{m^2}{eE_0} \gg 1; \\ G\left(\frac{d}{2}, \pi \frac{m^2}{eE_0}\right) &\approx \frac{2}{d}, \quad \delta \approx \frac{\sqrt{\pi}\Gamma(d/2)}{\Gamma(d/2 + 1/2)}, \quad m^2/eE_0 \ll 1. \end{aligned} \quad (2.21)$$

Let us turn to the vacuum-to-vacuum transition probability P_v defined by Eq. (A16) in Appendix A. It is given by similar forms for the peak field, the Sauter-like, and the T -constant field, respectively, with the corresponding N^{cr} [16, 22]:

$$\begin{aligned} P_v &= \exp(-\mu N^{\text{cr}}), \quad \mu = \sum_{l=0}^{\infty} \frac{\epsilon_{l+1}}{(l+1)^{d/2}} \exp\left(-l\pi \frac{m^2}{eE_0}\right), \\ \epsilon_l &= G\left(\frac{d}{2}, l\pi \frac{m^2}{eE_0}\right) \left[G\left(\frac{d}{2}, \pi \frac{m^2}{eE_0}\right)\right]^{-1} \text{ for a peak field,} \\ \epsilon_l &= \epsilon_l^{\text{T}} = 1 \text{ for } T\text{-constant field,} \\ \epsilon_l &= \epsilon_l^{\text{S}} = \delta^{-1} \sqrt{\pi} \Psi\left(\frac{1}{2}, \frac{2-d}{2}; l\pi \frac{m^2}{eE_0}\right) \text{ for Sauter-like fields.} \end{aligned} \quad (2.22)$$

In the case of a weak field ($m^2/eE_0 > 1$), $\epsilon_l^{\text{S}} \approx l^{-1/2}$ for the Sauter-like field, $\epsilon_l \approx l^{-1}$ for the peak field, and $\exp(-\pi m^2/eE_0) \ll 1$. Then $\mu \approx 1$ for all the cases in Eq. (2.22) and we have an universal relation $N^{\text{cr}} \approx \ln P_v^{-1}$. In the case of a strong field ($m^2/eE_0 \ll 1$), all the terms with different ϵ_l^{S} and ϵ_l contribute significantly to the sum in Eq. (2.22) if $l\pi m^2/eE_0 \lesssim 1$, and the quantities μ for Sauter-like and peak fields differ essentially from the case of T -constant field. Consequently, in this situation, one cannot derive an universal

relation between N^{cr} and P_v from particular cases given by Eq. (2.22). In addition, it should be noted that in the case of a strong field, when known semiclassical approaches are not applicable, the probability P_v (unlike the total number N^{cr}) has no a direct relation to vacuum mean values of the above discussed physical quantities. Therefore, to study an universal behavior of the vacuum instability in slowly varying strong electric fields one should derive first an universal form for the total density \tilde{n}^{cr} .

III. UNIVERSAL BEHAVIOR OF THE VACUUM INSTABILITY IN SLOWLY VARYING STRONG ELECTRIC FIELDS

A. Total density of created pairs

If the electric field is not very strong, mean numbers N_n^{cr} of created pairs (or distributions) at the final time instant are exponentially small, $N_n^{\text{cr}} \ll 1$. In this case the probability of the vacuum to remain a vacuum and probabilities of particle scattering and pair creation have simple representations in terms of these numbers,

$$|w_n(+ - | 0)|^2 \approx N_n^{\text{cr}}, \quad |w_n(- - | -)|^2 \approx (1 + N_n^{\text{cr}}), \quad P_v \approx 1 - \sum_n N_n^{\text{cr}}. \quad (3.1)$$

The latter relations are often used in semiclassical calculations to find N_n^{cr} and the total number of created pair $N^{\text{cr}} = \sum_n N_n^{\text{cr}}$ from the representation of P_v given by Schwinger's effective action.

However, when the electric field cannot be considered as a weak one (e.g. in some situations in astrophysics and condensed matter), the mean numbers N_n^{cr} can achieve their limited values $N_n^{\text{cr}} \rightarrow 1$ already at finite time instants t and the sum N^{cr} cannot be considered as a small quantity. Moreover, for slowly varying strong electric fields this sum is proportional to the large parameter $T_{\text{eff}}/\Delta t_{\text{st}}$. In such a case relations (3.1) are not correct anymore. However, as shown next, for arbitrary slowly varying strong electric field one can derive in the leading-term approximation an universal form for the total density of created pairs.

Let us define the range $D(t)$ as follows:

$$D(t) : \langle P_x(t) \rangle < 0, \quad |\langle P_x(t) \rangle| \gg \pi_{\perp}. \quad (3.2)$$

In this range the longitudinal kinetic momentum $\langle P_x(t) \rangle = p_x - U(t)$ is negative and big enough. If p_x components of the particle momentum belongs to the range $D(t)$, then the

particle energy is in main determined by an increment of the longitudinal kinetic momentum, $U(t) - U(t_{\text{in}})$, during the time interval $t - t_{\text{in}}$ and $\langle P_x(t) \rangle = \langle P_x(t_{\text{in}}) \rangle - [U(t) - U(t_{\text{in}})]$. Note that $D(t) \subset D(t')$ if $t < t'$. The leading term of the total number density of created pairs, \tilde{n}^{cr} , is formed over the range $D(t_{\text{out}})$, that is, the range $D(t_{\text{out}})$ is chosen as a realization of the subrange Ω in Eq. (2.16).

In the case when the electric field does not switch abruptly on and off, that is, the field slowly weakens at $t \rightarrow \pm\infty$ and one of the time instants t_{in} and t_{out} , or both are infinite $t_{\text{in}} \rightarrow -\infty$ and $t_{\text{out}} \rightarrow \infty$, one can ignore exponentially small contributions to \tilde{n}^{cr} from the time intervals $(t_{\text{in}}, t_{\text{in}}^{\text{eff}}]$ and $(t_{\text{out}}^{\text{eff}}, t_{\text{out}})$, where electric fields are much less than the maximum field E_0 , $E(t_{\text{in}}^{\text{eff}}), E(t_{\text{out}}^{\text{eff}}) \ll E_0$. Thus, in the general case it is enough to consider a finite interval $(t_{\text{in}}^{\text{eff}}, t_{\text{out}}^{\text{eff}}]$. Denoting $t_1 = t_{\text{in}}^{\text{eff}}$ and $t_{M+1} = t_{\text{out}}^{\text{eff}}$, we divide this interval into M intervals $\Delta t_i = t_{i+1} - t_i > 0$, $i = 1, \dots, M$, $\sum_{i=1}^M \Delta t_i = t_{\text{out}}^{\text{eff}} - t_{\text{in}}^{\text{eff}}$. We suppose that Eqs. (2.1) and (2.2) hold true for all the intervals, respectively. That allows us to treat the electric field as approximately constant within each interval, $\overline{E}(t) \approx \overline{E}(t_i)$, for $t \in (t_i, t_{i+1}]$. Note that inside of each interval Δt_i abrupt changes of the electric field $E(t)$ whose duration is much less than Δt_i , cannot change significantly the total value of \tilde{n}^{cr} , since $N_n^{\text{cr}} \leq 1$ for fermions. Using Eqs. (2.17) and (2.18) for the case of T -constant field, we can represent \tilde{n}^{cr} as the following sum

$$\tilde{n}^{\text{cr}} = \sum_{i=1}^M \Delta \tilde{n}_i^{\text{cr}}, \quad \Delta \tilde{n}_i^{\text{cr}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{e\overline{E}(t_i)}^{e\overline{E}(t_i)(t_i+\Delta t_i)} dp_x \int_{\sqrt{\lambda_i} < K_{\perp}} d\mathbf{p}_{\perp} N_n^{(i)},$$

$$N_n^{(i)} = e^{-\pi\lambda_i}, \quad \lambda_i = \frac{\pi_{\perp}^2}{e\overline{E}(t_i)}, \quad (3.3)$$

where K_{\perp} is any given number satisfying the condition $\sqrt{e\overline{E}(t_i)\Delta t_i} \gg K_{\perp}^2 \gg \max\{1, m^2/e\overline{E}(t_i)\}$. Taking into account Eq. (3.2), we represent the variable p_x as follows

$$p_x = U(t), \quad U(t) = \int_{t_{\text{in}}}^t dt' eE(t') + U(t_{\text{in}}). \quad (3.4)$$

Then neglecting small contributions to integral (3.3), we find the following universal form for the total density of created pairs in the leading-term approximation for arbitrary slowly varying strong electric field

$$\tilde{n}^{\text{cr}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^{t_{\text{out}}} dt eE(t) \int d\mathbf{p}_{\perp} N_n^{\text{uni}}, \quad N_n^{\text{uni}} = \exp\left[-\pi \frac{\pi_{\perp}^2}{eE(t)}\right]. \quad (3.5)$$

Note that N_n^{uni} is written in an universal form which can be used to calculate any total characteristics of the pair creation effect. After the integration over \mathbf{p}_\perp , we finally obtain

$$\tilde{n}^{\text{cr}} = \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^{t_{\text{out}}} dt [eE(t)]^{d/2} \exp \left\{ -\pi \frac{m^2}{eE(t)} \right\}. \quad (3.6)$$

These universal forms can be derived for bosons as well, if to restrict forms of external electric fields. Namely, by fields that have no abrupt variations of $E(t)$ that can produce significant grow of N_n^{cr} on a finite time interval. In fact, in this case we have to include in the range $D(t)$ the only subranges where $N_n^{\text{cr}} \leq 1$. In this case the universal forms for bosons are the same (3.5) and (3.6) assuming that $J_{(d)}$ is the number of the boson spin degrees of freedom, in particular, $J_{(d)} = 1$ for scalar particles and $J_{(4)} = 3$ for vector particles.

Using the identity $-\ln(1 - N_n^{\text{uni}}) = N_n^{\text{uni}} + (N_n^{\text{uni}})^2 \dots$, in the same manner one can derive an universal form of the vacuum-to-vacuum transition probability P_v defined for fermions by Eq. (A16) in Appendix A. First, we write

$$P_v \approx \exp \left[-\frac{V_{(d-1)} J_{(d)}}{(2\pi)^{d-1}} \sum_{l=1}^{\infty} \int_{t_{\text{in}}}^{t_{\text{out}}} dt eE(t) \int d\mathbf{p}_\perp (N_n^{\text{uni}})^l \right]. \quad (3.7)$$

Then, performing the integration over \mathbf{p}_\perp , we obtain that for fermions this universal form reads

$$P_v \approx \exp \left\{ -\frac{V_{(d-1)} J_{(d)}}{(2\pi)^{d-1}} \sum_{l=1}^{\infty} \int_{t_{\text{in}}}^{t_{\text{out}}} dt \frac{[eE(t)]^{d/2}}{l^{d/2}} \exp \left[-\pi \frac{lm^2}{eE(t)} \right] \right\}. \quad (3.8)$$

Taking into account that universal forms of \tilde{n}^{cr} for bosons are given by formulas similar to Eqs. (3.5) and (3.6) and using the definition of the vacuum-to-vacuum transition probability $P_v^{(\text{boson})}$ for bosons obtained in Refs. [3, 13],

$$P_v^{(\text{boson})} = \exp \left[-\sum_n \ln(1 + N_n^{\text{cr}}) \right], \quad (3.9)$$

we finally get in the Bose case the following universal form

$$P_v^{(\text{boson})} \approx \exp \left\{ -\frac{V_{(d-1)} J_{(d)}}{(2\pi)^{d-1}} \sum_{l=1}^{\infty} \int_{t_{\text{in}}}^{t_{\text{out}}} dt (-1)^{l-1} \frac{[eE(t)]^{d/2}}{l^{d/2}} \exp \left[-\pi \frac{lm^2}{eE(t)} \right] \right\}, \quad (3.10)$$

where $J_{(d)}$ is the number of boson spin degrees of freedom.

Using Eqs. (3.6) and (3.8), one obtains precisely expressions (2.18) and (2.22) that are found for the total densities and the vacuum-to-vacuum transition probabilities in the exactly solvable cases. Comparing Eqs. (3.6) and (3.10) with the exact results obtained for bosons

[16, 22], one find precise agreement too. Thus, we have an independent confirmation of the universal forms obtained above.

One can see that the obtained universal forms have specially simple forms in two limited cases, for a weak electric field ($m^2/eE_0 \gg 1$), when the term $[eE(t)]^{d/2}$ can be approximated by its maximal value $[eE_0]^{d/2}$, and for a strong enough electric field ($m^2/eE_0 \ll 1$), when there exist time intervals where $m^2/eE(t) \ll 1$ and approximations of the type

$$\exp \left[-\frac{\pi l m^2}{eE(t)} \right] = 1 - \frac{\pi l m^2}{eE(t)} + \dots \quad (3.11)$$

are available. Consider, for example, the case of a strong Gauss pulse,

$$E(t) = E_0 \exp \left[- (t/T_G)^2 \right], \quad (3.12)$$

with a large parameter $T_G \rightarrow \infty$. In this case we do not have an exact solution of the Dirac equation and known semiclassical approximations are not applicable. However, using approximation (3.11), we find from Eqs. (3.6) and (3.8) the leading terms as

$$\tilde{n}^{\text{cr}} \approx \frac{J_{(d)}(eE_0)^{d/2} T_G}{d(2\pi)^{d-2}}, \quad P_v \approx \exp \left[-V_{(d-1)} \tilde{n}^{\text{cr}} \sum_{l=1}^{\infty} l^{-d/2} \right]. \quad (3.13)$$

The representations (3.8) and (3.10) coincide with the leading term approximation of derivative expansion results from a field-theoretic calculations obtained in Refs. [6, 7] for $d = 3$ and $d = 4$. In this approximation the probability P_v was derived from a formal expansion in increasing numbers of derivatives of the background field strength for Schwinger's effective action:

$$S = S^{(0)}[F_{\mu\nu}] + S^{(2)}[F_{\mu\nu}, \partial_\mu F_{\nu\rho}] + \dots \quad (3.14)$$

where $S^{(0)}$ involves no derivatives of the background field strength $F_{\mu\nu}$, while the first correction $S^{(2)}$ involves two derivatives of the field strength, and so on, see Ref. [5] for a review. It was found that

$$P_v = \exp \left(-2\text{Im} S^{(0)} \right). \quad (3.15)$$

In the derivative expansion the fields are assumed to vary very slowly and satisfy the condition (2.1). A very convenient formalism for doing such an expansion is the worldline formalism, see [25] for the review, in which the effective action is written as a quantum mechanical path integral.

However, for a general background field, it is extremely difficult to estimate and compare the magnitude of various terms in the derivative expansion. Only under the assumption $m^2/eE_0 > 1$, one can demonstrate that the derivative expansion is completely consistent with the semiclassical WKB analysis of the imaginary part of the effective action [26]. It is shown only for a constant electric field that Eq. (3.15) is given exactly by the semiclassical WKB limit when the leading order of fluctuations is taken into account [27].

It should be stressed that unlike to the authors of Refs. [6, 7], we derive Eqs. (3.8) and (3.10) in the framework of the general exact formulation of strong-field QED [3, 13], where P_v are defined by Eqs. (A16) and (3.9), respectively. Therefore we obtain Eqs. (3.8) and (3.10) independently from the derivative expansion approach and the obtained result holds true for any strong field under consideration. Thus, it is proven that Eq. (3.15) is given exactly by the semiclassical WKB limit for arbitrary slowly varying electric field.

B. Evolution of vacuum instability

In this section details of the time evolution of vacuum instability effects are of interest. In particular, the study of the time evolution of the mean electric current, energy, and momentum provides us with new characteristics of the effect, related, in particular, with the back reaction. Due to the translational invariance of the uniform external field, the all corresponding mean values are proportional to the space volume. Therefore, it is enough to calculate the vacuum mean values of the current density vector $\langle j^\mu(t) \rangle$ and of the EMT $\langle T_{\mu\nu}(t) \rangle$, defined by Eq. (A17), see Appendix A. Note that these densities depend on the initial vacuum, on the evolution of the electric field from the initial time instant up to the current time instant t , but they do not depend on the further history of the system and definition of particle-antiparticle at the time t .

Let us consider the time dependence of the current density vector $\langle j^\mu(t) \rangle$ and of the EMT $\langle T_{\mu\nu}(t) \rangle$, given by Eqs. (A23). Due to the uniform character of the distributions N_n^{cr} , the only diagonal matrix elements of EMT differ from zero, in particular, for $d \neq 3$ the only longitudinal current components are not zero. In $d = 3$ dimensions, there are two non-equivalent representations for γ -matrices, $\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^2$, $\gamma^2 = -i(\pm 1)\sigma^1$, where σ^i are Pauli matrices, and representations with the sign $+$ or $-$ in the round brackets correspond to different fermion species, the so-called $+$ and $-$ fermions, respectively. Due to this fact,

a non-zero current component $\langle j^2(t) \rangle$ can exist, this fact is related to the so-called Chern-Simons term in the effective action [28, 29], see details in Ref. [8]. However, if there are both fermion species in a model, as it takes place, for example, in the Dirac model of the graphene, then $\langle j^2(t) \rangle = 0$.

It follows from Eqs. (A22) and (A23) that the nonzero terms $\text{Re}\langle j^\mu(t) \rangle^p$ and $\text{Re}\langle T_{\mu\nu}(t) \rangle^p$ appear due to the vacuum instability. These terms are growing with time due to an increase of the number of states that are occupied by created pairs. In any system of Fermi particles the mean value $\langle j^2(t) \rangle$ is finite.

As a consequence of Eq. (3.2), we have

$$i\partial_t {}^\pm\varphi_n(t) \approx \pm |\langle P_x(t) \rangle| {}^\pm\varphi_n(t), \quad (3.16)$$

which means that at the time t we deal with an ultrarelativistic particle and its kinetic momentum $\langle P_x(t) \rangle$ can be considered as a large parameter. Considering time dependence of means $\text{Re}\langle j^1(t) \rangle^p$ and $\text{Re}\langle T_{\mu\mu}(t) \rangle^p$, we suppose that the time difference $t - t_{\text{in}}$ is big enough to satisfy Eq. (3.16). Using exact relation Eq. (A10) to express solutions ${}^\pm\psi_n$ via ${}^\pm\varphi_n$, and neglecting strongly oscillating terms, we find that leading contribution to the function $S^p(x, x')$ (defined by Eq. (A22)) at $t \sim t'$ can be represented by the following expression

$$S^p(x, x') \approx -i \sum_n N_n^{\text{cr}} \left[{}^+\psi_n(x) {}^+\bar{\psi}_n(x') - {}^-\psi_n(x) {}^-\bar{\psi}_n(x') \right]. \quad (3.17)$$

It is clear that for any large enough difference $t - t_{\text{in}}$ the integral over momentum p in the right hand side of Eq. (3.17) can be approximated by an integral over the range $D(t_{\text{out}})$ that gives the dominant contribution to the mean number of created particles with respect to the total increment of the longitudinal kinetic momentum. Moreover, taking into account Eqs. (3.2) and (3.4), we see that $D(t) \subset D(t') \subset D(t_{\text{out}})$ if $t < t' < t_{\text{out}}$ and for a given difference $t - t_{\text{in}}$ the dominant contribution to the right hand side of Eq. (3.17) is from a subrange $D(t) \subset D(t_{\text{out}})$.

We recall that, according to Eq. (A4), one can choose the corresponding in- and out-Dirac solutions either with $\chi = +1$ or with $\chi = -1$. Using this possibility, we choose $\chi = +1$ for ${}^+\psi_n(x)$ and $\chi = -1$ for ${}^-\psi_n(x)$. With such a choice, taking into account that $\mathbf{p} \in D(t)$, we simplify essentially the matrix structure of the representation (3.17). Thus, after a summation over spin polarizations σ , we obtain the following result:

$$S^p(x, x') \approx (\gamma P + m) \Delta^p(x, x'), \quad (3.18)$$

where the function $\Delta^p(x, x')$ reads

$$\begin{aligned} \Delta^p(x, x') = & -i \sum_{\mathbf{p} \in D(t)} N_n^{\text{cr}} |\langle P_x(t) \rangle| \exp[i\mathbf{p}(\mathbf{r} - \mathbf{r}')] \\ & \times \left\{ (1 + \gamma^0 \gamma^1) \left[{}^+ \varphi_n(t) {}^+ \varphi_n^*(t') \right] \Big|_{\chi=+1} + (1 - \gamma^0 \gamma^1) \left[{}^- \varphi_n(t) {}^- \varphi_n^*(t') \right] \Big|_{\chi=-1} \right\}. \end{aligned}$$

Using Eq. (3.18) in Eq. (A23), we find the following representations for the vacuum means of current density and EMT components:

$$\begin{aligned} \langle j^1(t) \rangle^p & \approx 2e \frac{V_{(d-1)} J_{(d)}}{(2\pi)^{d-1}} \int_{\mathbf{p} \in D(t)} d\mathbf{p} N_n^{\text{cr}} \rho(t) |\langle P_x(t) \rangle|; \\ \langle T_{00}(t) \rangle^p & \approx \langle T_{11}(t) \rangle^p \approx \frac{V_{(d-1)} J_{(d)}}{(2\pi)^{d-1}} \int_{\mathbf{p} \in D(t)} d\mathbf{p} N_n^{\text{cr}} \rho(t) \langle P_x(t) \rangle^2, \\ \langle T_{ll}(t) \rangle^p & \approx \frac{V_{(d-1)} J_{(d)}}{(2\pi)^{d-1}} \int_{\mathbf{p} \in D(t)} d\mathbf{p} N_n^{\text{cr}} \rho(t) p_l^2, \quad l = 2, \dots, D, \\ \rho(t) & = 2 |\langle P_x(t) \rangle| \left\{ \left| {}^+ \varphi_n(t) \right|^2 \Big|_{\chi=+1} + \left| {}^- \varphi_n(t) \right|^2 \Big|_{\chi=-1} \right\}. \end{aligned} \quad (3.19)$$

One can verify, taken into account Eq. (3.16), that the functions ${}^\zeta \varphi_n(t)$ can be approximated by their asymptotics (A6) in the range $D(t)$ if the instant value of the longitudinal kinetic momentum differs slightly at the time instant t from its final value, such that

$$|\langle P_x(t_{\text{out}}) \rangle - \langle P_x(t) \rangle| \ll |\langle P_x(t) \rangle|. \quad (3.20)$$

In a sense this means that the time instant t is close enough to the final time instant, $t \rightarrow t_{\text{out}}$. We find that

$$\rho(t)|_{t \rightarrow t_{\text{out}}} = [V_{(d-1)} |\langle P_x(t_{\text{out}}) \rangle|]^{-1}. \quad (3.21)$$

Then taken into account Eq. (2.16), we obtain from Eq. (3.19) that

$$\langle j^1(t) \rangle^p \Big|_{t \rightarrow t_{\text{out}}} \approx 2e \tilde{n}^{\text{cr}}, \quad (3.22)$$

where \tilde{n}^{cr} is given by Eqs. (3.5) and (3.6). It means that dominant contributions to the mean numbers N_n^{cr} of created particles are formed before the time instant t that satisfies Eq. (3.20). For $t > t_{\text{out}}$, the pair production stops, vacuum polarization effects disappear, and quantities (3.19) for $t > t_{\text{out}}$ maintain their values at $t = t_{\text{out}}$. Using Eq. (3.21), we obtain that

$$\begin{aligned} \langle j^1(t) \rangle \Big|_{t > t_{\text{out}}} & \approx \langle j^1(t) \rangle^p \Big|_{t \rightarrow t_{\text{out}}} \approx 2e \tilde{n}^{\text{cr}}, \\ \langle T_{00}(t) \rangle \Big|_{t > t_{\text{out}}} & \approx \langle T_{11}(t) \rangle \Big|_{t > t_{\text{out}}} \approx \langle T_{11}(t) \rangle^p \Big|_{t \rightarrow t_{\text{out}}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{\mathbf{p} \in D(t)} d\mathbf{p} N_n^{\text{cr}} |\langle P_x(t_{\text{out}}) \rangle|, \\ \langle T_{ll}(t) \rangle \Big|_{t > t_{\text{out}}} & \approx \langle T_{ll}(t) \rangle^p \Big|_{t \rightarrow t_{\text{out}}} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{\mathbf{p} \in D(t)} d\mathbf{p} N_n^{\text{cr}} |\langle P_x(t_{\text{out}}) \rangle|^{-1} p_l^2, \quad l = 2, \dots, D. \end{aligned} \quad (3.23)$$

Using the universal form of the differential number of created pairs, $N_n^{\text{cr}} \approx N_n^{\text{uni}}$, given by Eq. (3.5), making variable change (3.4), and performing the integration over p_\perp , we finally obtain from Eq. (3.23) the following new result: at the final time instant EMT components have the following universal behavior:

$$\begin{aligned} \langle T_{00}(t) \rangle|_{t>t_{\text{out}}} &\approx \langle T_{11}(t) \rangle|_{t>t_{\text{out}}} \approx \langle T_{11}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} \\ &\approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^{t_{\text{out}}} dt [U(t_{\text{out}}) - U(t)] [eE(t)]^{d/2} \exp \left[-\pi \frac{m^2}{eE(t)} \right], \\ \langle T_{ll}(t) \rangle|_{t>t_{\text{out}}} &\approx \langle T_{ll}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} \approx \frac{J_{(d)}}{(2\pi)^d} \int_{t_{\text{in}}}^{t_{\text{out}}} \frac{dt [eE(t)]^{d/2+1}}{[U(t_{\text{out}}) - U(t)]} \exp \left[-\pi \frac{m^2}{eE(t)} \right]. \end{aligned} \quad (3.24)$$

The quantity $\langle T_{00}(t) \rangle|_{t>t_{\text{out}}}$ is the mean energy density of pairs created at any time instant t with zero longitudinal kinetic momentum and then accelerated to final longitudinal kinetic momenta from zero to its maximum ΔU . The quantity $\langle T_{11}(t) \rangle|_{t>t_{\text{out}}}/2\tilde{n}^{\text{cr}}$ is the mean kinetic momentum per particle at the time instant t_{out} . The energy density $\langle T_{00}(t) \rangle|_{t>t_{\text{out}}}$ is equal to the pressure $\langle T_{11}(t) \rangle|_{t>t_{\text{out}}}$ along the direction of the electric field at the time instant t_{out} . This equality is a natural equation of state for noninteracting particles accelerated by an electric field to relativistic velocities.

In particular, for fields admitting exactly solvable cases (these fields are given by Eqs. (2.4), (2.8), and (2.10)), we find from Eq. (3.24):

a) For T -constant field:

$$\begin{aligned} \langle T_{00}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} &\approx \langle T_{11}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} \approx eE_0 r^{\text{cr}} (t_{\text{out}} - t_{\text{in}})^2, \\ \langle T_{ll}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} &\approx \pi^{-1} r^{\text{cr}} \ln \left[\sqrt{eE_0} (t_{\text{out}} - t_{\text{in}}) \right], \quad l = 2, \dots, D. \end{aligned} \quad (3.25)$$

b) For the peak field:

$$\begin{aligned} \langle T_{00}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} &\approx \langle T_{11}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} \approx eE_0 r^{\text{cr}} [k_2^{-1} + k_1^{-1}] \\ &\times \left\{ [k_2^{-1} - k_1^{-1}] G \left(\frac{d}{2} + 1, \frac{\pi m^2}{eE_0} \right) + k_1^{-1} G \left(\frac{d}{2}, \frac{\pi m^2}{eE_0} \right) \right\}, \\ \langle T_{ll}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} &\approx \frac{r^{\text{cr}}}{2\pi} \left[G \left(\frac{d}{2} - 1, \frac{\pi m^2}{eE_0} \right) + \frac{k_2}{k_1} G \left(\frac{d}{2}, \frac{\pi m^2}{eE_0} \right) \right], \quad l = 2, \dots, D. \end{aligned} \quad (3.26)$$

Densities (3.26) correspond to the case of an exponentially decaying field as $k_1^{-1} \rightarrow 0$.

c) For Sauter-like field:

$$\begin{aligned} \langle T_{00}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} &\approx \langle T_{11}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} \approx eE_0 r^{\text{cr}} T_S^2 \left[\delta - G \left(\frac{d}{2}, \frac{\pi m^2}{eE_0} \right) \right], \\ \langle T_{ll}(t) \rangle^p|_{t \rightarrow t_{\text{out}}} &\approx \frac{r^{\text{cr}}}{2\pi} \left[\sqrt{\pi} \Psi \left(\frac{1}{2}, 2 - \frac{d}{2}; \frac{\pi m^2}{eE_0} \right) + G \left(\frac{d}{2} - 1, \frac{\pi m^2}{eE_0} \right) \right]. \end{aligned} \quad (3.27)$$

Note that using the differential mean numbers of created pairs given by Eqs. (2.6), (2.7), and (2.13) for the exactly solvable cases, we obtain from Eq. (3.23) literally expressions (3.25) (earlier obtained in Refs. [8, 20]), (3.26), and (3.27). It is an independent confirmation of universal form (3.24). We stress that Eqs.(3.26) and (3.27) are first obtained in this article.

It should be noted that the densities $\langle j^1(t) \rangle^p|_{t \rightarrow t_{\text{out}}}$ and $\langle T_{\mu\mu}(t) \rangle^p|_{t \rightarrow t_{\text{out}}}$ are formed over all the time interval $t_{\text{out}} - t_{\text{in}}$ of the field duration. All these densities are growing functions of the increment of the longitudinal kinetic momentum. However, they differ, in particular, because switching on and off conditions of the corresponding electric fields are different.

In what follows we show that some universal behavior of the densities $\langle j^1(t) \rangle^p$ and $\langle T_{\mu\mu}(t) \rangle^p$ can be derived from general forms (3.19) for any large difference $t - t_{\text{in}}$, even if $t - t_{\text{in}} \ll t_{\text{out}} - t_{\text{in}}$. We begin the demonstration of this fact with the case of a finite interval of time when the electric field potential can be approximated by a potential of a constant electric field Eq. (2.3). At the same time, we assume that $\langle P_x(t) \rangle$ satisfies condition (3.2) at the time t . It is convenient to compare the cases of T -constant and exponentially decaying fields, which both are abruptly switching on but their ways of switching off may be different.

In the case of exponentially decaying field, the functions ${}^{\pm}\varphi_n(t)$ in Eq. (3.19) are given by the second line in Eq. (2.11) and approximation (2.3) holds if $k_2 t \ll 1$. Then $|\langle P_x(t) \rangle| \ll |\pi_2|$. To obtain functions ${}^{\pm}\varphi_n(t)$ in such an approximation we use first the asymptotic representation for the confluent hypergeometric function $\Phi(a, c; \eta)$ via the Weber parabolic cylinder functions (WPCFs) for large η and c with fixed a and $\tau = \eta/c \sim 1$, given by Eq. (13.8.4) in [30]. Assuming then $|\tau - 1| \sim 1$ and using asymptotic expansions of WPCFs one finds $\Phi(a, c; \eta) \approx (1 - \tau)^{-a}$ for $1 - \tau > 0$. Thus, we obtain

$$\rho(t) = [V_{(d-1)} |\langle P_x(t) \rangle|]^{-1}. \quad (3.28)$$

In the range $D(t)$, the distribution N_n^{cr} is approximately given by Eq. (2.7). Finally we obtain

$$\begin{aligned} \langle j^1(t) \rangle^p &\approx 2er^{\text{cr}}\Delta t, \quad \Delta t = t - t_{\text{in}}, \\ \langle T_{00}(t) \rangle^p &\approx \langle T_{11}(t) \rangle^p \approx eE_0 r^{\text{cr}} \Delta t^2, \\ \langle T_{ll}(t) \rangle^p &\approx \pi^{-1} r^{\text{cr}} \ln \left(\sqrt{eE_0} \Delta t \right) \quad \text{if } l = 2, \dots, D, \end{aligned} \quad (3.29)$$

where $\Delta t = t - t_{\text{in}}$ is the duration time of a constant field. In this case $t_{\text{in}} = 0$.

The field potential of the T -constant field (2.8) has the form (2.3) in the intermediate region II. For sufficiently large times $t < t_{\text{out}}$, when the longitudinal kinetic momentum belongs to the range $D(t)$, the distribution N_n^{cr} is approximately given by Eq. (2.7). In this case, exact expressions for the functions $^+\varphi_n(t)$, see Eq. (26) in Ref. [16], and similar expressions for the functions $^-\varphi_n(t)$ can be approximated as the following Weber parabolic cylinder functions (WPCFs):

$$\begin{aligned} ^+\varphi_n(t) &\approx V_{(d-1)}^{-1/2} CD_{-1-\rho}[(1+i)\xi], \quad ^-\varphi_n(t) \approx V_{(d-1)}^{-1/2} CD_{\rho}[(1-i)\xi], \\ \xi &= (eE_0 t - p_x)(eE_0)^{-1/2}, \quad C = (2eE_0)^{-1/2} \exp(-\pi\lambda_0/8). \end{aligned} \quad (3.30)$$

Then we find from Eq. (3.19) that the densities $\langle j^1(t) \rangle^p$ and $\langle T_{\mu\mu}(t) \rangle^p$ have the same form (3.29) with $t_{\text{in}} = -T/2$.

Note that the above results are obtained by using functions $^{\pm}\varphi_n(t)$, which have in and out-asymptotics at t_{out} . Nevertheless, these results show also that densities (3.29) are not affected by evolution of the functions $^{\pm}\varphi_n(t)$ from t to t_{out} in the range $p \in D(t)$, assuming that the corresponding electric field exists during a macroscopically large time period Δt , satisfying Eq. (2.1). This fact is closely related with a characteristic property of the kernel of integrals (3.19), which will be derived from an universal form of the total density of created pairs given by Eq. (3.5). Let $t'_{\text{out}} < t_{\text{out}}$ is another possible final time instant, then

$$\tilde{n}^{\text{cr}}(t'_{\text{out}}) \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^{t'_{\text{out}}} dt [eE(t)]^{d/2} \exp \left\{ -\pi \frac{m^2}{eE(t)} \right\} \quad (3.31)$$

Eq. (3.31) corresponds to the assumption that in the range $p \in D(t'_{\text{out}}) \subset D(t_{\text{out}})$ the electric field is switched on at t_{in} and switched off at t'_{out} . Then instead of functions $^{\zeta}\psi_n(x)$ satisfying the eigenvalue problem (A3), we have to use solutions of the following eigenvalue problem

$$H(t) ^{\zeta}\psi_n^{(t'_{\text{out}})}(x) = ^{\zeta}\varepsilon_n ^{\zeta}\psi_n^{(t'_{\text{out}})}(x), \quad t \in [t'_{\text{out}}, +\infty), \quad ^{\zeta}\varepsilon_n = ^{\zeta}p_0(t'_{\text{out}}).$$

Using the representation

$$^{\zeta}\psi_n^{(t'_{\text{out}})}(x) = [i\partial_t + H(t)] \gamma^0 \exp(i\mathbf{p}\mathbf{r}) ^{\zeta}\varphi_n^{(t'_{\text{out}})}(t) v_{\chi,\sigma}$$

we obtain

$$\begin{aligned} ^{\zeta}\varphi_n^{(t'_{\text{out}})}(t) &= ^{\zeta}N^{(t'_{\text{out}})} \exp[-i^{\zeta}p_0(t'_{\text{out}})(t - t'_{\text{out}})], \quad t \in [t'_{\text{out}}, +\infty), \\ ^{\zeta}N^{(t'_{\text{out}})} &= (2p_0(t'_{\text{out}}) \{p_0(t'_{\text{out}}) - \chi\zeta[p_x - U(t'_{\text{out}})]\} V_{(d-1)})^{-1/2}. \end{aligned} \quad (3.32)$$

Thus, leading contribution to the function $S^p(x, x')$ (defined by Eq. (A22)) at $t' \sim t < t'_{\text{out}}$ can be expressed via $\zeta\psi_n^{(t'_{\text{out}})}(x)$ as follows

$$S^p(x, x') \approx -i \sum_{\sigma, \mathbf{p} \in D(t)} N_n^{\text{cr}} \left[+\psi_n^{(t'_{\text{out}})}(x) + \bar{\psi}_n^{(t'_{\text{out}})}(x') - -\psi_n^{(t'_{\text{out}})}(x) - \bar{\psi}_n^{(t'_{\text{out}})}(x') \right]. \quad (3.33)$$

Then $\rho(t)$ in Eq. (3.19) can be represented as

$$\rho(t) = 2 |\langle P_x(t) \rangle| \left\{ \left| +\varphi_n^{(t'_{\text{out}})}(t) \right|_{\chi=+1}^2 + \left| -\varphi_n^{(t'_{\text{out}})}(t) \right|_{\chi=-1}^2 \right\}.$$

Taken into account Eq. (3.32), we can see that Eq. (3.28) holds for any large time difference $t - t_{\text{in}}$. Using the universal form of the differential numbers of created pairs, $N_n^{\text{cr}} \approx N_n^{\text{uni}}$, given by Eq. (3.5), changing the variable according to Eq. (3.4), and performing the integration over p_{\perp} , we find from Eq. (3.19) that the vacuum mean values of current and EMT components have the following universal behavior for any large difference $t - t_{\text{in}}$:

$$\begin{aligned} \langle j^1(t) \rangle^p &\approx 2e\tilde{n}^{\text{cr}}(t), \\ \langle T_{00}(t) \rangle^p &\approx \langle T_{11}(t) \rangle^p \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{t_{\text{in}}}^t dt' [U(t) - U(t')] [eE(t')]^{d/2} \exp \left[-\frac{\pi m^2}{eE(t')} \right], \\ \langle T_{ll}(t) \rangle^p &\approx \frac{J_{(d)}}{(2\pi)^d} \int_{t_{\text{in}}}^t \frac{dt' [eE(t')]^{d/2+1}}{[U(t) - U(t')]} \exp \left[-\frac{\pi m^2}{eE(t')} \right], \quad l = 2, \dots, D. \end{aligned} \quad (3.34)$$

Here $\tilde{n}^{\text{cr}}(t)$ is given by Eq. (3.31). In particular, when $t = t_{\text{out}}$ one obtains Eq. (3.22) and (3.24).

The obtained results show that the scale $\Delta t_{\text{st}}^{\text{m}}$ plays the role of the stabilization time for the densities $\langle j^1(t) \rangle^p$ and $\langle T_{\mu\mu}(t) \rangle^p$. The characteristic parameter m^2/eE_0 can be represented as the ratio of two characteristic lengths: $c^3 m^2/\hbar e E_0 = (c\Delta t_{\text{st}}/\Lambda_C)^2$, where $\Lambda_C = \hbar/mc$ is the Compton wavelength. In strong electric fields, $(c\Delta t_{\text{st}}/\Lambda_C)^2 \lesssim 1$, inequality (2.2) is simplified to the form $\Delta t/\Delta t_{\text{st}} \gg 1$, in which the Compton wavelength is absent. We see that the scale Δt_{st} plays the role of the stabilization time for a strong electric field. This means that Δt_{st} is a characteristic time scale which allows us to distinguish fields that have microscopic or macroscopic time change, it plays similar role as the Compton wavelength plays in the case of a weak field. Therefore, calculations in a T -constant field are quite representative for a large class of slowly varying electric fields.

In which follows we use the example of the T -constant field to consider the contributions $\text{Re}\langle j^{\mu}(t) \rangle^c$ and $\text{Re}\langle T_{\mu\nu}(t) \rangle^c$ to the mean values of the current density $\langle j^{\mu}(t) \rangle$ and the EMT

$\langle T_{\mu\nu}(t) \rangle$, given by Eqs. (A23). Note that the mean current density $\langle j^\mu(t) \rangle$ and the physical part of the mean value $\langle T_{\mu\nu}(t) \rangle$ are zero for any $t < t_{\text{in}}$. For $t > t_{\text{in}}$, we are interested in these mean values only for a large time periods $\Delta t = t - t_{\text{in}}$ satisfying Eqs. (2.2) and (2.3). In this case, the longitudinal kinetic momentum belongs to the range (3.2) and distributions N_n^{cr} are approximated by Eq. (2.7). Using approximation (3.30), the functions $_{-}\varphi_n(t)$, given by Eq. (25) in Ref. [16], and similar functions $_{+}\varphi_n(t)$ can be taken in the following form

$$_{-}\varphi_n(t) = V_{(d-1)}^{-1/2} C D_{-1-\rho} [-(1+i)\xi], \quad _{+}\varphi_n(t) = V_{(d-1)}^{-1/2} C D_{\rho} [-(1-i)\xi]. \quad (3.35)$$

In the same approximation, the causal propagator $S^c(x, x')$ (A21) can be calculated using solutions $^{\pm}\psi_n(x)$ and $_{\pm}\psi_n(x)$ with scalar functions given by Eqs. (3.30) and (3.35) in the range (3.2). It can be shown that the main contributions to $\text{Re} \langle j^\mu(t) \rangle^c$, $\langle j^2(t) \rangle$ and $\text{Re} \langle T_{\mu\mu}(t) \rangle^c$ are formed in the range (3.2) for a large time period Δt . It is important that these contributions are independent of the interval Δt , that is, the densities $\text{Re} \langle j^\mu(t) \rangle^c$, $\langle j^2(t) \rangle$, and $\text{Re} \langle T_{\mu\mu}(t) \rangle^c$ are local quantities describing only vacuum polarization effects. Then we integrate in Eq. (A21) over all the momenta. Thus, we see that in the case under consideration, the propagator $S^c(x, x')$ can be approximated by the propagator in a constant uniform electric field.

The propagator $S^c(x, x')$ in a constant uniform electric field can be represented as the Fock–Schwinger proper time integral:

$$S^c(x, x') = (\gamma P + m) \Delta^c(x, x'), \quad \Delta^c(x, x') = \int_0^\infty f(x, x', s) ds, \quad (3.36)$$

see [17] and [31], where the Fock–Schwinger kernel $f(x, x', s)$ reads

$$f(x, x', s) = \exp \left(i \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu} s \right) f^{(0)}(x, x', s), \quad f^{(0)}(x, x', s) = - \frac{e E_0 s^{-d/2+1}}{(4\pi i)^{d/2} \sinh(e E_0 s)} \\ \times \exp \left[-i (e \Lambda + m^2 s) + \frac{1}{4i} (x - x') e F \coth(e F s) (x - x') \right].$$

Here $\coth(e F s)$ is the matrix with the components $[\coth(e F s)]^\mu{}_\nu$, $F_{\mu\nu} = E_0 (\delta_\mu^0 \delta_\nu^1 - \delta_\mu^1 \delta_\nu^0)$, and $\Lambda = (t + t')(x_1 - x'_1) E_0 / 2$, see [4, 32].

It is easy to see that $\langle j^1(t) \rangle^c = 0$, as should be expected due to the translational symmetry. If $d = 3$ there is a transverse vacuum-polarization current,

$$\langle j^2(t) \rangle = \pm \frac{e^2}{4\pi^{3/2}} \gamma \left(\frac{1}{2}, \frac{\pi m^2}{e E_0} \right) E_0, \quad (3.37)$$

for each \pm fermion, [8] (see, as well, Ref. [33]), where $\gamma(1/2, x)$ is the incomplete gamma function. Note that the transverse current of created particles is absent, $\langle j^2(t) \rangle = 0$ if $t > t_{\text{out}}$. The factor in the front of E_0 in Eq. (3.37) can be considered as a nonequilibrium Hall conductivity for large duration of the electric field. In the presence of both \pm fermions in a model, $\langle j^2(t) \rangle = 0$ for any t .

Using Eq. (3.36), we obtain components of the EMT for the T -constant field in the following form

$$\begin{aligned} \text{Re}\langle T_{00}(t) \rangle^c &= -\text{Re}\langle T_{11}(t) \rangle^c = E_0 \frac{\partial \text{Re}\mathcal{L}[E_0]}{\partial E_0} - \text{Re}\mathcal{L}[E_0] , \\ \text{Re}\langle T_{ll}(t) \rangle^c &= \text{Re}\mathcal{L}[E_0] , \quad l = 2, \dots, D, \end{aligned} \quad (3.38)$$

where

$$\mathcal{L}[E_0] = \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{tr} f(x, x, s), \quad \text{tr} f(x, x, s) = 2^{[d/2]} \cosh(eE_0 s) f^{(0)}(x, x, s). \quad (3.39)$$

The quantity $\mathcal{L}[E_0]$ can be identified with a non-renormalized one-loop effective Euler-Heisenberg Lagrangian of the Dirac field in an uniform constant electric field E_0 . Note that components $\text{Re}\langle T_{\mu\nu}(t) \rangle^c$ do not depend of the time duration Δt of the T -constant field if Δt is sufficiently large.

This result can be generalized to the case of arbitrary slowly varying electric field. To this end we divide as before the finite interval $(t_{\text{in}}^{\text{eff}}, t_{\text{out}}^{\text{eff}}]$ into M intervals $\Delta t_i = t_{i+1} - t_i > 0$, such that Eqs. (2.1) and (2.2) hold true for each of them. That allows us to treat the electric field as approximately constant within each interval, $\overline{E(t)} \approx \overline{E}(t_i)$ for $t \in (t_i, t_{i+1}]$. In each such an interval, we obtain expressions similar to the ones (3.38) and (3.39), where the electric field E_0 has to be substituted by $\overline{E}(t_i)$. Then components of the EMT for arbitrary slowly varying strong electric field $E(t)$ in the leading-term approximation can be represented as

$$\begin{aligned} \text{Re}\langle T_{00}(t) \rangle^c &= -\text{Re}\langle T_{11}(t) \rangle^c = E(t) \frac{\partial \text{Re}\mathcal{L}[E(t)]}{\partial E(t)} - \text{Re}\mathcal{L}[E(t)] , \\ \text{Re}\langle T_{ll}(t) \rangle^c &= \text{Re}\mathcal{L}[E(t)] , \quad l = 2, \dots, D, \end{aligned} \quad (3.40)$$

where

$$\begin{aligned} \mathcal{L}[E(t)] &= \frac{1}{2} \int_0^\infty \frac{ds}{s} \text{tr} \tilde{f}(x, x, s), \quad \text{tr} \tilde{f}(x, x, s) = 2^{[d/2]} \cosh[eE(t)s] \tilde{f}^{(0)}(x, x, s), \\ \tilde{f}^{(0)}(x, x, s) &= -\frac{eE(t) s^{-d/2+1} \exp(-im^2 s)}{(4\pi i)^{d/2} \sinh[eE(t)s]}. \end{aligned} \quad (3.41)$$

Note that $\mathcal{L}[E(t)]$ evolves in time due to the time dependence of the field $E(t)$.

The quantity $\mathcal{L}[E(t)]$ describes the vacuum polarization. The quantities (3.40) are divergent due to the real part of the effective Lagrangian (3.41), which is ill defined. This real part must be regularized and renormalized. In low dimensions, $d \leq 4$, $\text{Re}\mathcal{L}[E(t)]$ can be regularized in the proper-time representation and renormalized by the Schwinger renormalizations of the charge and the electromagnetic field [4]. In particular, for $d = 4$, the renormalized effective Lagrangian $\mathcal{L}_{ren}[E(t)]$ is

$$\mathcal{L}_{ren}[E(t)] = \int_0^\infty \frac{ds \exp(-im^2 s)}{8\pi^2 s} \left\{ \frac{eE(t) \coth[eE(t)s]}{s} - \frac{1}{s^2} - \frac{[eE(t)s]^2}{3} \right\}. \quad (3.42)$$

In higher dimensions, $d > 4$, a different approach is required. One can give a precise meaning and calculate the one-loop effective action using zeta-function regularization, see details in Ref. [8]. If we are interested in the case of a very strong field, $m^2/eE(t) \ll 1$ then

$$\text{Re}\mathcal{L}_{ren}[E(t)] \sim \begin{cases} [eE(t)]^{d/2}, & d \neq 4k \\ [eE(t)]^{d/2} \ln[eE(t)/M^2], & d = 4k, k \in \mathbb{N} \end{cases}, \quad (3.43)$$

where the quantity M is a renormalization scale. In the framework of the on-shell renormalization of a massive theory, we have to set $M = m$. Making the same renormalization for $\langle T_{\mu\mu}(t) \rangle^c$, we can see that for the renormalized EMT the following relations hold true

$$\begin{aligned} \text{Re}\langle T_{00}(t) \rangle_{ren}^c &= -\text{Re}\langle T_{11}(t) \rangle_{ren}^c = E(t) \frac{\partial \text{Re}\mathcal{L}_{ren}[E(t)]}{\partial E(t)} - \text{Re}\mathcal{L}_{ren}[E(t)], \\ \text{Re}\langle T_{ll}(t) \rangle_{ren}^c &= \text{Re}\mathcal{L}_{ren}[E(t)], \quad l = 2, 3, \dots, D. \end{aligned} \quad (3.44)$$

In the strong-field case, the leading contributions to the renormalized EMT are

$$\text{Re}\langle T_{\mu\mu}(t) \rangle_{ren}^c \sim \begin{cases} [eE(t)]^{d/2}, & d \neq 4k \\ [eE(t)]^{d/2} \ln[eE(t)/M^2], & d = 4k \end{cases}. \quad (3.45)$$

The final form of the vacuum mean components of the EMT are

$$\langle T_{\mu\mu}(t) \rangle_{ren} = \text{Re}\langle T_{\mu\mu}(t) \rangle_{ren}^c + \text{Re}\langle T_{\mu\mu}(t) \rangle^p, \quad (3.46)$$

where the components $\text{Re}\langle T_{\mu\mu}(t) \rangle_{ren}^c$ and $\text{Re}\langle T_{\mu\mu}(t) \rangle^p$ are given by Eqs. (3.44) and (3.34), respectively. For $t < t_{in}$ and $t > t_{out}$ the electric field is absent such that $\text{Re}\langle T_{\mu\mu}(t) \rangle_{ren}^c = 0$.

In the right hand side of Eq. (3.46), the term $\text{Re}\langle T_{\mu\mu}(t) \rangle^p$ represents contributions due to the vacuum instability, whereas the term $\text{Re}\langle T_{\mu\mu}(t) \rangle_{ren}^c$ represents vacuum polarization

effects. For weak electric fields, $m^2/eE_0 \gg 1$, contributions due to the vacuum instability are exponentially small, so that the vacuum polarization effects play the principal role. For strong electric fields, $m^2/eE_0 \ll 1$, the energy density of the vacuum polarization $\text{Re} \langle T_{00}(t) \rangle_{ren}^c$ is negligible compared to the energy density due to the vacuum instability $\langle T_{00}(t) \rangle^p$,

$$\langle T_{\mu\mu}(t) \rangle_{ren} \approx \text{Re} \langle T_{\mu\mu}(t) \rangle^p. \quad (3.47)$$

The latter density is formed on the whole time interval $t - t_{in}$, however, dominant contributions are due to time intervals Δt_i with $m^2/e\bar{E}(t_i) < 1$ and the large dimensionless parameters $\sqrt{e\bar{E}(t_i)}\Delta t_i$.

We note that effective Lagrangian (3.41) and its renormalized form $\mathcal{L}_{ren}[E(t)]$ coincide with leading term approximation of derivative expansion results from field-theoretic calculations obtained in Refs. [6, 7] for $d = 3$ and $d = 4$. In this approximation the $S^{(0)}$ term of the Schwinger's effective action, given by the expansion (3.14), has the form

$$S^{(0)}[F_{\mu\nu}] = \int dx \mathcal{L}_{ren}[E(t)]. \quad (3.48)$$

It should be stressed that unlike to the authors of Refs. [6, 7], we derive Eq. (3.41) and its renormalized form in the framework of the general exact formulation of strong-field QED [3, 13], using QED definition of the mean value of the EMT, given by Eq. (A23). Therefore we obtain $\mathcal{L}_{ren}[E(t)]$ independently from the derivative expansion approach and the obtained result holds true for any strong field under consideration. Moreover, it is proven that in this case not only the imaginary part of $S^{(0)}$ but its real part as well is given exactly by the semiclassical WKB limit. It is clearly demonstrated that the imaginary part of the effective action, $\text{Im}S^{(0)}$, is related to the vacuum-to-vacuum transition probability P_v and can be represented as an integral of $\mathcal{L}_{ren}[E(t)]$ over the total field history, whereas the kernel of the real part of this effective action, $\text{Re}\mathcal{L}_{ren}[E(t)]$, is related to the local EMT which defines the vacuum polarization. Obtained results justify the derivative expansion as an asymptotic expansion that can be useful to find the corrections for mean values of the EMT components.

In what follows considering example of the T -constant field we demonstrate that under natural assumptions, the parameter $eE_0\Delta t^2$ is limited. For $d > 4$ an exact meaning of finite terms of the effective Lagrangian (3.39) can be understood only from the corresponding fundamental theory. Considering problems of high-energy physics in $d = 4$, it is usually assumed that just from the beginning there exists an uniform classical electric field with a

given energy density. The system of particles interacting with this field is closed, that is, the total energy of the system is conserved. Under such an assumption, the pair creation is a transient process and the applicability of the constant field approximation is limited by the smallness of the back reaction, which implies the following restriction from above:

$$(\Delta t / \Delta t_{\text{st}})^2 \ll \frac{\pi^2}{J\alpha} \exp\left(\pi \frac{c^3 m^2}{\hbar e E_0}\right), \quad (3.49)$$

on time Δt for a given electric field strength. Here α is the fine structure constant and J is the number of the spin degrees of freedom, see [19]. Thus, there is a range of the parameters E_0 and Δt where the approximation of the constant external field is consistent. For QCD with a constant $SU(3)$ chromoelectric field E_0^a ($a = 1, \dots, 8$) (during the period when the produced partons can be treated as weakly coupled due to the property of asymptotic freedom in QCD), and at low temperatures $\theta \ll q\sqrt{C_1}\Delta t$, the consistency restriction for the dimensionless parameter $q\sqrt{C_1}\Delta t^2$ has the form $1 \ll q\sqrt{C_1}\Delta t^2 \ll \pi^2/3q^2$, where q is the coupling constant and $C_1 = E_0^a E_0^a$ is a Casimir invariant for $SU(3)$.

The case of $d = 3$ attracts attention in recent years. It is well known that at certain conditions (the so-called charge neutrality point) electronic excitations in graphene monolayer behave as relativistic Dirac massless fermions in $2 + 1$ dimensions, with the Fermi velocity $v_F \simeq 10^6$ m/s playing the role of the speed of light, see details in recent reviews [34, 35]. Then in the range of the applicability of the Dirac model to the graphene physics, any electric field is strong. There appears a time scale specific to graphene (and to similar nanostructures with the Dirac fermions), $\Delta t_{\text{st}}^g = (eE_0 v_F / \hbar)^{-1/2}$, which plays the role of the stabilization time in the case under consideration. The generation of a mass gap in the graphene band structure is an important fundamental and practical problem under current research. In the presence of the mass gap $\Delta\varepsilon = mv_F^2$, the stabilization condition has a general form

$$\Delta t / \Delta t_{\text{st}}^{\text{gm}} \gg 1, \quad \Delta t_{\text{st}}^{\text{gm}} = \Delta t_{\text{st}}^g \max\{1, (\Delta\varepsilon)^2 / \hbar v_F e E_0\}. \quad (3.50)$$

In this case, the strong field condition reads $(\Delta\varepsilon)^2 / v_F \hbar e E_0 \ll 1$. It is shown [9] that the time scale Δt_{st}^g appears for the tight-binding model as the time scale when the perturbation theory with respect to the electric field breaks down, and the dc response changes from the linear-in- E_0 duration-independent regime to a nonlinear-in- E_0 and duration-dependent regime. In the experimental situation described in Ref. [10], a constant voltage between two electrodes connected to the graphene was applied, $V = E_0 L_x$, and current-voltage

characteristics ($I - V$) are measured within ~ 1 s, which is a very large time scale compared with the ballistic flight time $T_{bal} = L_x/v_F$ for a finite flake length L_x . In typical experiments, $L_x \sim 1 \mu\text{m}$, so that $T_{bal} \sim 10^{-12}$ s. To match our results with these conditions, our time Δt should be replaced by some typical time-scale that we call the effective time duration Δt_{eff} . In the absence of the dissipation, the transport is ballistic; in this case, considering a strip with a lateral infinite width, we assume the ballistic flight time T_{bal} to be the effective time duration, $\Delta t_{eff} = T_{bal}$. In a realistic sample, placed on a substrate, the effective time duration Δt_{eff} can be many times smaller than T_{bal} , because of charged impurities or structural disorder of the substrate. However, such an effective time Δt_{eff} remains macroscopically large, so that Eq. (3.50) still holds. The external constant electric field can be considered as a good approximation of the effective mean field as long as the field produced by the induced current of created particles is negligible compared to the applied field. Then $\langle j^1(t) \rangle^p$ in Eq. (3.29) describes a regime where the current behaves as $j \sim V^{3/2}$. An experimental observation of this $I - V$ was recently reported for low-mobility samples (the case $\Delta t_{eff} \ll T_{bal}$) [10]. This implies the consistency restriction $\Delta t \ll \Delta t_{br} = \Delta t_{st}^g/4\alpha$ [8]. Thus, there is a window in the parameter range of E_0 and Δt where the model with constant external field is consistent. For example, let us assume that $\Delta t \sim T_{bal}$. It implies that $7 \times 10^{-4} \text{ V} \ll V \ll 8 \text{ V}$. These voltages are in the range typically used in experiments with the graphene.

IV. CONCLUDING REMARKS

In the present article, we have revised vacuum instability effects in three exactly solvable cases in QED with t -electric potential steps that have real physical importance. These are Sauter-like electric field, the so-called T -constant electric field and exponentially growing and decaying strong electric fields in a slowly varying regime. Defining the slowly varying regime in general terms, we can observe the existence of universal forms for the time evolution of vacuum effects caused by strong electric field. Such universality appears when the duration of the external field is sufficiently large in comparison to the scale $\Delta t_{st} = \left[e\overline{E(t)} \right]^{-1/2}$. In this case the scale of the time varying for an external field and leading contributions to vacuum mean values are macroscopic. Here, we find universal approximate representations for the total density of created pairs and vacuum means of current density and EMT com-

ponents that hold true for arbitrary t -electric potential step slowly varying with time. These representations do not require knowledge of corresponding solutions of the Dirac equation, they have a form of simple functionals of a given slowly varying electric field. We establish relations of these representations with leading term approximations of derivative expansion results. These results allow on to formulate some semiclassical approximations that are not restricted by smallness of differential mean numbers of created pairs. We have tested the obtained representations in the cases of exactly solvable t -electric potential steps. For time instants t close enough to the final time t_{out} , $t \rightarrow t_{\text{out}}$, the leading vacuum characteristics are formed due to real pair production. One can say that we have isolated global contributions that depend on the total history of an electric field. Current density and EMT components of created pairs for T -constant electric field can be easily extracted from the above mentioned representations. In such a way components of Sauter-like field and exponentially growing and decaying fields for $t \rightarrow t_{\text{out}}$ are obtained for the first time. All these densities are growing functions of the increment of the longitudinal kinetic momentum. However, their explicit forms differ, in particular, since switching on and off conditions of electric fields are different. It should be also noted that an universal behavior of the vacuum mean current and EMT components was discovered for time intervals, inside of which the electric field potential can be approximated by a potential of a constant electric field. We see that for such time intervals components of vacuum means of current density and EMT can be divided in global contributions, dependent on the effective time duration of the electric field, Δt , and independent on switching off manner, and local contributions that do not depend on the interval Δt and are function of slowly varying electric field, $E(t)$. The global contributions define equations of state for the matter field, which is a plasma of some kind of electron-positron excitations created from vacuum. We show that local components of vacuum mean EMT can be expressed via the one-loop effective Euler-Heisenberg Lagrangian of the Dirac field and satisfy an equation of state for electromagnetic field.

The reason for the universal behavior in the case under consideration is the following: For total physical quantities as current density and EMT of created pairs a large effective time of the field duration corresponds to a large density of states that are occupied by created pairs if an electric field is strong enough. One can guess that the universality under the question is associated with the big state density that is a large parameter in the slowly varying regime. Technically, we take into account only leading terms with respect to this large

parameter terms, whereas oscillation terms are disregarded. In fact, in this approximation the pair creation can be treated as a phase transition from the initial vacuum to a plasma of electron-positron pairs.

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Appendix A: Vacuum instability in QED with t -electric potential steps

Nonperturbative approach to the $d = D + 1$ -dimensional model of Dirac fields interacting with a strong t -electric potential steps is based on the complete sets of exact solutions of the Dirac equation. Potentials $A^\mu(x)$, $x = (x^\mu) = (x^0 = t, \mathbf{r})$, $\mathbf{r} = (x^i)$ of external electromagnetic fields² corresponding to t -electric potential steps are defined as

$$A^0 = 0, \quad \mathbf{A}(t) = (A^1 = A_x(t), \quad A^l = 0, \quad l = 2, \dots, D), \quad A_x(t) \xrightarrow{t \rightarrow \pm\infty} A_x(\pm\infty), \quad (\text{A1})$$

where $A_x(\pm\infty)$ are some constant quantities, and the time derivative of the potential $A_x(t)$ does not change its sign for any $t \in \mathbb{R}$. For definiteness, it is supposed that $\dot{A}_x(t) \leq 0 \implies A_x(-\infty) > A_x(+\infty)$. We stress that homogeneous electric fields under consideration $\mathbf{E}(t) = (E_x(t), 0, \dots, 0)$ are switched off as $|t| \rightarrow \infty$, $E_x(t) = -\dot{A}_x(t) = E(t) \geq 0$, $E(t) \xrightarrow{|t| \rightarrow \infty} 0$.

The Dirac equation reads

$$\begin{aligned} i\partial_t \psi(x) &= H(t) \psi(x), \quad H(t) = \gamma^0 (\boldsymbol{\gamma} \mathbf{P} + m), \\ P_x &= -i\partial_x - U(t), \quad \mathbf{P}_\perp = -i\nabla_\perp, \quad U(t) = -eA_x(t), \end{aligned} \quad (\text{A2})$$

where $H(t)$ is the one-particle Dirac Hamiltonian; $\psi(x)$ is a $2^{[d/2]}$ -component spinor; $[d/2]$ stands for the integer part of $d/2$; $m \neq 0$ is the electron mass; the index \perp stands for components of the momentum operator that are perpendicular to the electric field. Here,

² The Greek indexes span the Minkowsky space-time, $\mu = 0, 1, \dots, D$, and the Latin indexes span the Euclidean space, $l = 1, \dots, D$. We use the system of units where $\hbar = c = 1$.

γ^μ are the γ -matrices in d dimensions [36]. The number of spin degree of freedom is $J_{(d)} = 2^{[d/2]-1}$. We choose the electron as the main particle with the charge $-e$, where $e > 0$ is the absolute value of the electron charge.

The quantization of the Dirac field in the background under consideration is based on the existence of solutions to the Dirac equation with special asymptotics as $t \rightarrow \pm\infty$. For instance, we let the electric field be switched on at t_{in} and switched off at t_{out} , so that the interaction between the Dirac field and the electric field vanishes at all time instants outside the interval $t \in (t_{\text{in}}, t_{\text{out}})$. We choose that before time t_{in} and after time t_{out} the spinors $\psi_n(x)$, $n = (\mathbf{p}, \sigma)$, are states with a definite momentum $\mathbf{p} = (p_x, \mathbf{p}_\perp)$ and spin polarization $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{[d/2]-1})$, $\sigma_s = \pm 1$, and satisfy the following eigenvalue problems:

$$\begin{aligned} H(t) \, {}_\zeta\psi_n(x) &= {}_\zeta\varepsilon_n \, {}_\zeta\psi_n(x) \, , \quad t \in (-\infty, t_{\text{in}}] \, , \quad {}_\zeta\varepsilon_n = \zeta p_0(t_{\text{in}}) \, , \\ H(t) \, {}^\zeta\psi_n(x) &= {}^\zeta\varepsilon_n \, {}^\zeta\psi_n(x) \, , \quad t \in [t_{\text{out}}, +\infty) \, , \quad {}^\zeta\varepsilon_n = \zeta p_0(t_{\text{out}}) \, , \\ p_0(t) &= \sqrt{[p_x - U(t)]^2 + \pi_\perp^2} \, , \quad \pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2} . \end{aligned} \quad (\text{A3})$$

where the additional quantum number $\zeta = \pm$ labels states, respectively. In these asymptotic states, $\zeta = +$ correspond to free electrons and $\zeta = -$ corresponds to free positrons.

In what follows, we consider two complete sets ${}_\zeta\psi_n(x)$ and ${}^\zeta\psi_n(x)$ of solutions of Dirac equation (A2) (in- and out solutions respectively),

$$\begin{aligned} {}_\zeta\psi_n(x) &= [i\partial_t + H(t)] \gamma^0 \exp(i\mathbf{p}\mathbf{r}) \, {}_\zeta\varphi_n(t) v_{\chi,\sigma} , \\ {}^\zeta\psi_n(x) &= [i\partial_t + H(t)] \gamma^0 \exp(i\mathbf{p}\mathbf{r}) \, {}^\zeta\varphi_n(t) v_{\chi,\sigma} . \end{aligned} \quad (\text{A4})$$

Here $v_{\chi,\sigma}$ is a set of constant orthonormalized spinors, $\gamma^0\gamma^1 v_{\chi,\sigma} = \chi v_{\chi,\sigma}$, $\chi = \pm 1$, $v_{\chi,\sigma}^\dagger v_{\chi',\sigma'} = \delta_{\chi,\chi'} \delta_{\sigma,\sigma'}$. In dimensions $d > 3$, one can subject the spinors v_χ to some supplementary conditions determining spin polarization σ_s [in the dimensions $d = 2, 3$ there are no spin degrees of freedom that are described by the quantum numbers σ], and, together with the additional index χ , provide a convenient parametrization of the solutions. The scalar functions ${}_\zeta\varphi_n(t)$ and ${}^\zeta\varphi_n(t)$ obey the second order differential equation

$$\left\{ \frac{d^2}{dt^2} + [p_x - U(t)]^2 + \pi_\perp^2 - i\chi\dot{U}(t) \right\} \begin{pmatrix} {}_\zeta\varphi_n(t) \\ {}^\zeta\varphi_n(t) \end{pmatrix} = 0 . \quad (\text{A5})$$

In the asymptotic regions

$$\begin{aligned} {}_\zeta\varphi_n(t) &= {}_\zeta N \exp[-i {}_\zeta\varepsilon_n(t - t_{\text{in}})] \, , \quad t \in (-\infty, t_{\text{in}}] \, , \\ {}^\zeta\varphi_n(t) &= {}^\zeta N \exp[-i {}^\zeta\varepsilon_n(t - t_{\text{out}})] \, , \quad t \in [t_{\text{out}}, +\infty) \, , \end{aligned} \quad (\text{A6})$$

where ${}_{\zeta}N$, ${}^{\zeta}N$, are normalization constants, and there exists an energy gap between the electron and positron states. Since χ is not a physical quantum number (the spin operator $\gamma^0\gamma^1$ does not commute with the Dirac Hamiltonian (A2) in case $m \neq 0$), we select the same χ for each ${}_{\zeta}\psi_n(x)$ and ${}^{\zeta}\psi_n(x)$. Solutions (A4) are subject to the orthonormality conditions (the standard volume regularization with a large spatial box of volume $V_{(d-1)}$ is used). Then

$$\begin{aligned} ({}_{\zeta}\psi_n, {}_{\zeta'}\psi'_{n'}) &= \delta_{n,n'}\delta_{\zeta,\zeta'} , \quad \left({}^{\zeta}\psi_n, {}^{\zeta'}\psi'_{n'} \right) = \delta_{n,n'}\delta_{\zeta,\zeta'} , \\ {}_{\zeta}\mathcal{N} &= \left[2p_0(t_{\text{in}}) q_{\text{in}}^{\zeta} V_{(d-1)} \right]^{-1/2} , \quad {}^{\zeta}\mathcal{N} = \left[2p_0(t_{\text{out}}) q_{\text{out}}^{\zeta} V_{(d-1)} \right]^{-1/2} \end{aligned} \quad (\text{A7})$$

where $q_{\text{in/out}}^{\zeta} = p_0(t_{\text{in/out}}) - \chi\zeta[p_x - U(t_{\text{in/out}})]$. The inner products $({}_{\zeta'}\psi_{n'}, {}^{\zeta}\psi_n)$ are diagonal in quantum numbers n and n' ,

$$({}_{\zeta'}\psi_{n'}, {}^{\zeta}\psi_n) = \delta_{n',n} g({}_{\zeta'}|\zeta) , \quad g({}_{\zeta'}|\zeta) = g(\zeta|{}_{\zeta'})^* . \quad (\text{A8})$$

The corresponding diagonal matrix elements g obey the unitarity relations

$$\sum_{\varkappa} g(\zeta|\varkappa) g(\varkappa|{}_{\zeta'}) = \sum_{\varkappa} g(\zeta|\varkappa) g(\varkappa|{}_{\zeta'}) = \delta_{\zeta,\zeta'} \quad (\text{A9})$$

and relate in- and out-solutions $\{{}_{\zeta}\psi_n(x)\}$ and $\{{}^{\zeta}\psi_n(x)\}$ for each n ,

$$\begin{aligned} {}^{\zeta}\psi_n(x) &= g(+|\zeta) {}_+\psi_n(x) + g(-|\zeta) {}_-\psi_n(x) , \\ {}_{\zeta}\psi_n(x) &= g(+|\zeta) {}_+\psi_n(x) + g(-|\zeta) {}_-\psi_n(x) . \end{aligned} \quad (\text{A10})$$

Decomposing the Dirac operator $\hat{\Psi}(x)$ in the complete sets of in- and out-solutions [3, 13],

$$\hat{\Psi}(x) = \sum_n [a_n(\text{in}) {}_+\psi_n(x) + b_n^{\dagger}(\text{in}) {}_-\psi_n(x)] = \sum_n [a_n(\text{out}) {}_+\psi_n(x) + b_n^{\dagger}(\text{out}) {}_-\psi_n(x)] , \quad (\text{A11})$$

we introduce in- and out-creation and annihilation Fermi operators. Their nonzero anticommutation relations are,

$$[a_n(\text{in}), a_m^{\dagger}(\text{in})]_+ = [a_n(\text{out}), a_m^{\dagger}(\text{out})]_+ = [b_n(\text{in}), b_m^{\dagger}(\text{in})]_+ = [b_n(\text{out}), b_m^{\dagger}(\text{out})]_+ = \delta_{nm} . \quad (\text{A12})$$

In these terms, the Heisenberg Hamiltonian is diagonalized at $t \leq t_{\text{in}}$ and $t \geq t_{\text{out}}$,

$$\begin{aligned} \hat{\mathbb{H}}(t) &= \sum_n \{ {}_+\varepsilon_n a_n^+(\text{in}) a_n(\text{in}) + |{}_-\varepsilon_n| b_n^+(\text{in}) b_n(\text{in}) \} , \quad t \leq t_{\text{in}} , \\ \hat{\mathbb{H}}(t) &= \sum_n \{ {}_+\varepsilon_n a_n^+(\text{out}) a_n(\text{out}) + |{}_-\varepsilon_n| b_n^+(\text{out}) b_n(\text{out}) \} , \quad t \geq t_{\text{out}} , \end{aligned} \quad (\text{A13})$$

where the diverging c-number parts have been omitted, as usual. The initial $|0, \text{in}\rangle$ and final $|0, \text{out}\rangle$ vacuum vectors, as well as many-particle in- and out-states, are defined by

$$\begin{aligned} a_n(\text{in})|0, \text{in}\rangle &= b_n(\text{in})|0, \text{in}\rangle = 0, \quad a_n(\text{out})|0, \text{out}\rangle = b_n(\text{out})|0, \text{out}\rangle = 0, \\ |\text{in}\rangle &= b_n^+(\text{in})\dots a_n^+(\text{in})\dots|0, \text{in}\rangle, \quad |\text{out}\rangle = b_n^+(\text{out})\dots a_n^+(\text{out})\dots|0, \text{out}\rangle. \end{aligned} \quad (\text{A14})$$

Using the charge operator one can see that a_n^\dagger , a_n are the creation and annihilation operators of electrons, whereas b_n^\dagger , b_n are the creation and annihilation operators of positrons.

Transition amplitudes in the Heisenberg representation have the form $M_{\text{in} \rightarrow \text{out}} = \langle \text{out} | \text{in} \rangle$. In particular, the vacuum-to-vacuum transition amplitude reads $c_v = \langle 0, \text{out} | 0, \text{in} \rangle$. Relative probability amplitudes of particle scattering, pair creation and annihilation are:

$$\begin{aligned} w(+|+)_{n'n} &= c_v^{-1} \langle 0, \text{out} | a_{n'}(\text{out}) a_n^\dagger(\text{in}) | 0, \text{in} \rangle = \delta_{n,n'} w_n(+|+), \\ w(-|-)_{n'n} &= c_v^{-1} \langle 0, \text{out} | b_{n'}(\text{out}) b_n^\dagger(\text{in}) | 0, \text{in} \rangle = \delta_{n,n'} w_n(-|-), \\ w(+ - | 0)_{n'n} &= c_v^{-1} \langle 0, \text{out} | a_{n'}(\text{out}) b_n(\text{out}) | 0, \text{in} \rangle = \delta_{n,n'} w_n(+ - | 0), \\ w(0| - +)_{nn'} &= c_v^{-1} \langle 0, \text{out} | b_n^\dagger(\text{in}) a_{n'}^\dagger(\text{in}) | 0, \text{in} \rangle = \delta_{n,n'} w_n(0| - +). \end{aligned} \quad (\text{A15})$$

The in- and out-operators are related by linear canonical transformations,

$$a_n(\text{out}) = g(+|+)a_n(\text{in}) + g(+|-)b_n^\dagger(\text{in}), \quad b_n^\dagger(\text{out}) = g(-|+)a_n(\text{in}) + g(-|-)b_n^\dagger(\text{in}).$$

These relations allows one to calculate the differential mean numbers of electrons $N_n^a(\text{out})$ and positrons $N_n^b(\text{out})$ created from the vacuum state as

$$\begin{aligned} N_n^a(\text{out}) &= \langle 0, \text{in} | a_n^\dagger(\text{out}) a_n(\text{out}) | 0, \text{in} \rangle = |g(-|+)|^2, \\ N_n^b(\text{out}) &= \langle 0, \text{in} | b_n^\dagger(\text{out}) b_n(\text{out}) | 0, \text{in} \rangle = |g(+|-)|^2, \quad N_n^{\text{cr}} = N_n^b(\text{out}) = N_n^a(\text{out}). \end{aligned}$$

By N_n^{cr} we denote the differential numbers of created pairs. Relative probabilities (A15), the vacuum-to-vacuum transition amplitude c_v , the probability for a vacuum to remain a vacuum P_v as well as the total number N of pairs created from vacuum can be expressed via the distribution N_n^{cr} ,

$$\begin{aligned} |w_n(+ - | 0)|^2 &= N_n^{\text{cr}} (1 - N_n^{\text{cr}})^{-1}, \quad |w_n(-|-)|^2 = (1 - N_n^{\text{cr}})^{-1}, \\ P_v &= |c_v|^2 = \prod_n (1 - N_n^{\text{cr}}), \quad N^{\text{cr}} = \sum_n N_n^{\text{cr}} = \sum_n |g(-|+)|^2. \end{aligned} \quad (\text{A16})$$

The vacuum mean electric current, energy, and momentum are defined as integrals over the spatial volume. Due to the translational invariance in the uniform external field, all

these mean values are proportional to the space volume. Therefore, it is enough to calculate the vacuum mean values of the current density vector $\langle j^\mu(t) \rangle$ and of the energy-momentum tensor (EMT) $\langle T_{\mu\nu}(t) \rangle$, defined as

$$\langle j^\mu(t) \rangle = \langle 0, \text{in} | j^\mu | 0, \text{in} \rangle, \quad \langle T_{\mu\nu}(t) \rangle = \langle 0, \text{in} | T_{\mu\nu} | 0, \text{in} \rangle. \quad (\text{A17})$$

Here we stress the time dependence of mean values (A17), which does exist due to a time dependence of the external field. We recall for further convenience the form of the operators of the current density and the EMT of the quantum Dirac field,

$$\begin{aligned} j^\mu &= \frac{q}{2} \left[\bar{\hat{\Psi}}(x), \gamma^\mu \hat{\Psi}(x) \right], \quad T_{\mu\nu} = \frac{1}{2} (T_{\mu\nu}^{\text{can}} + T_{\nu\mu}^{\text{can}}), \\ T_{\mu\nu}^{\text{can}} &= \frac{1}{4} \left\{ [\bar{\hat{\Psi}}(x), \gamma_\mu P_\nu \hat{\Psi}(x)] + [P_\nu^* \bar{\hat{\Psi}}(x), \gamma_\mu \hat{\Psi}(x)] \right\}, \\ P_\mu &= i\partial_\mu - qA_\mu(x), \quad \bar{\hat{\Psi}}(x) = \hat{\Psi}^\dagger(x) \gamma^0. \end{aligned} \quad (\text{A18})$$

Note that the mean values (A17) depend on the definition of the initial vacuum, $|0, \text{in}\rangle$ and on the evolution of the electric field from the time t_{in} of switching it on up to the current time instant t , but they do not depend on the further history of the system. The renormalized vacuum mean values $\langle j^\mu(t) \rangle$ and $\langle T_{\mu\nu}(t) \rangle$, $t_{\text{in}} < t < t_{\text{out}}$ are sources in equations of motion for mean electromagnetic and metric fields, respectively. In particular, complete description of the back reaction is related to the calculation of these mean values for any t .

Mean values and probability amplitudes are calculated by the help of different kind of propagators. The probability amplitudes are calculated using Feynman diagrams with the causal (Feynman) propagator

$$S^c(x, x') = i \langle 0, \text{out} | \hat{T} \hat{\Psi}(x) \hat{\Psi}^\dagger(x') \gamma^0 | 0, \text{in} \rangle c_v^{-1}, \quad (\text{A19})$$

where \hat{T} denotes the chronological ordering operation. A perturbation theory (with respect to radiative processes) uses the so-called in-in propagator $S_{\text{in}}^c(x, x')$ and $S^p(x, x')$ propagator,

$$S_{\text{in}}^c(x, x') = i \langle 0, \text{in} | \hat{T} \hat{\Psi}(x) \hat{\Psi}^\dagger(x') \gamma^0 | 0, \text{in} \rangle, \quad S^p(x, x') = S_{\text{in}}^c(x, x') - S^c(x, x'). \quad (\text{A20})$$

All the above propagators can be expressed via the in- and out-solution as follows:

$$\begin{aligned} S^c(x, x') &= i \begin{cases} \sum_n {}^+ \psi_n(x) \omega_n(+|+) {}^+ \bar{\psi}_n(x'), & t > t' \\ - \sum_n {}^- \psi_n(x) \omega_n(-|-) {}^- \bar{\psi}_n(x'), & t < t' \end{cases}, \quad (\text{A21}) \\ S_{\text{in}}^c(x, x') &= i \begin{cases} \sum_n {}^+ \psi_n(x) {}^+ \bar{\psi}_n(x'), & t > t' \\ - \sum_n {}^- \psi_n(x) {}^- \bar{\psi}_n(x'), & t < t' \end{cases}, \quad S^p(x, x') = -i \sum_n {}^- \psi_n(x) \omega_n(0|-+) {}^+ \bar{\psi}_n(x'). \end{aligned} \quad (\text{A22})$$

The mean values of the operator (A18) are expressed via the latter propagators as

$$\begin{aligned}
\langle j^\mu(t) \rangle &= \text{Re} \langle j^\mu(t) \rangle^c + \text{Re} \langle j^\mu(t) \rangle^p, \quad \langle j^\mu(t) \rangle^{c,p} = iq \text{tr} [\gamma^\mu S^{c,p}(x, x')] |_{x=x'}, \\
\langle T_{\mu\nu}(t) \rangle &= \text{Re} \langle T_{\mu\nu}(t) \rangle^c + \text{Re} \langle T_{\mu\nu}(t) \rangle^p, \quad \langle T_{\mu\nu}(t) \rangle^{c,p} = i \text{tr} [A_{\mu\nu} S^{c,p}(x, x')] |_{x=x'}, \\
A_{\mu\nu} &= 1/4 [\gamma_\mu (P_\nu + P_\nu'^*) + \gamma_\nu (P_\mu + P_\mu'^*)] .
\end{aligned} \tag{A23}$$

Here tr stands for the trace in the γ -matrices indices and the limit $x \rightarrow x'$ is understood as follows:

$$\text{tr}[R(x, x')]_{x=x'} = \frac{1}{2} \left[\lim_{t \rightarrow t'-0} \text{tr}[R(x, x')] + \lim_{t \rightarrow t'+0} \text{tr}[R(x, x')] \right]_{\mathbf{x}=\mathbf{x}'},$$

where $R(x, x')$ is any two point matrix function.

The function $S^p(x, y)$ vanishes in the case of a stable vacuum. In this case and only in this case $\langle j^\mu(t) \rangle = \text{Re} \langle j^\mu(t) \rangle^c$, $\langle T_{\mu\nu}(t) \rangle = \text{Re} \langle T_{\mu\nu}(t) \rangle^c$.

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